Machine Learning I

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Machine Learning for Computer Vision
TU Dresden

https://mlcv.cs.tu-dresden.de/courses/24-winter/ml1/

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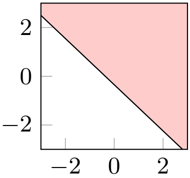
Contents. This part of the course introduces the notion of labeled data, the supervised learning problem, the separation problem, the separability problem and the inference problem.

Supervised learning



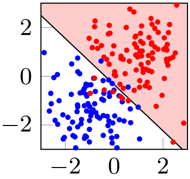
Example: A medical test with $n \in \mathbb{N}$ design parameters $\theta \in \Theta = \mathbb{R}^n$ measures $m \in \mathbb{N}$ quantities and indicates by $y \in Y = \{0, 1\}$ whether a measurement $x \in X = \mathbb{R}^m$ is considered to be healthy $(y = 0)$ or pathological $(y = 1)$.

$$X \xrightarrow{g_\theta} Y$$

Informally, **supervised learning** is the problem of finding, in a family $g : \Theta \to Y^X$, one function $g_\theta : X \to Y$ that minimizes a weighted sum of two objectives:

- g_θ deviates little from a finite set $\{(x_s, y_s)\}_{s \in S}$ of input-output-pairs, called **labeled data**
- g_θ has low complexity, as quantified by a function $R : \Theta \to \mathbb{R}_0^+$, called a **regularizer**

Supervised learning



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In this course, we concentrate exclusively on the special case where Y is finite. To begin with, we even concentrate on the case where $Y = \{0, 1\}$, i.e., learning how to make yes/no decisions. Hence, we consider a family $g\colon \Theta \to \{0, 1\}^X$.

We allow ourselves to take a detour by not optimizing over a family $g : \Theta \to \{0, 1\}^X$ directly but instead optimizing over a family $f : \Theta \to \mathbb{R}^X$ and defining g wrt. f via a function $L : \mathbb{R} \times \{0, 1\} \to \mathbb{R}_0^+$, called a **loss function**, such that

$$\forall \theta \in \Theta \ \forall x \in X\colon \quad g_\theta(x) \in \underset{\hat{y} \in \{0,1\}}{\operatorname{argmin}} \ L(f_\theta(x), \hat{y}) \ . \tag{1}$$

Example: 0-1-loss

$$\forall r \in \mathbb{R} \ \forall \hat{y} \in \{0, 1\}\colon \quad L(r, \hat{y}) = \begin{cases} 0 & r > 0 \wedge \hat{y} = 1 \\ 0 & r \leq 0 \wedge \hat{y} = 0 \\ 1 & \text{otherwise} \end{cases} \ . \tag{2}$$

Next, we define the supervised learning problem rigorously.

Definition. For any finite, non-empty set S, called a set of **samples**, any $X \neq \emptyset$, called an **feature space** and any $x : S \to X$, the tuple (S, X, x) is called **unlabeled data**.

For any $y : S \to \{0, 1\}$, given in addition and called a **labeling**, the tuple (S, X, x, y) is called **labeled data**.

Definition. For any labeled data $T = (S, X, x, y)$, any $\Theta \neq \emptyset$ and $f : \Theta \to \mathbb{R}^X$, any $R : \Theta \to \mathbb{R}_0^+$, called a **regularizer**, any $L : \mathbb{R} \times \{0,1\} \to \mathbb{R}_0^+$, called a **loss function**, and any $\lambda \in \mathbb{R}_0^+$:

- The instance of the **supervised learning problem** has the form

$$\inf_{\theta \in \Theta} \quad \lambda R(\theta) + \sum_{s \in S} L(f_\theta(x_s), y_s) \tag{3}$$

- The instance of the **separation problem** has the form

$$\begin{aligned} \inf_{\theta \in \Theta} \quad & R(\theta) && (4) \\ \text{subject to} \quad & \forall s \in S : \quad L(f_\theta(x_s), y_s) = 0 && (5) \end{aligned}$$

- The instance of the **separability problem** consists in deciding whether there exists a $\theta \in \Theta$ such that

$$\begin{aligned} R(\theta) &\leq m && (6) \\ \forall s \in S : \quad L(f_\theta(x_s), y_s) &= 0 && (7) \end{aligned}$$

Definition. For any unlabeled data $T = (S, X, x)$, any $\hat{f} : X \to \mathbb{R}$ and any $L : \mathbb{R} \times \{0, 1\} \to \mathbb{R}_0^+$, the instance of the **inference problem** wrt. $T, \hat{f}$ and L is defined as

$$\min_{y \in \{0,1\}^S} \sum_{s \in S} L(\hat{f}(x_s), y_s) \tag{8}$$

Lemma. The solutions to the inference problem are the $y : S \to \{0, 1\}$ such that

$$\forall s \in S\colon \quad y_s \in \operatorname*{argmin}_{\hat{y} \in \{0,1\}} \; L(\hat{f}(x_s), \hat{y}) \;\; . \tag{9}$$

Moreover, if $\hat{f}(X) \subseteq \{0, 1\}$ and L is the 0-1-loss, then

$$\forall s \in S\colon \quad y_s = \hat{f}(x_s) \;\; . \tag{10}$$

Summary.

- The **supervised learning problem** is an optimization problem. It consists in finding, in a family of functions, one function that minimizes a weighted sum of two objectives:
  1. It deviates little from given labeled data, as quantified by a loss function
  2. It has low complexity, as quantified by a regularizer.
- The **separation problem** is an optimization problem. It consists in finding a function in the family that minimizes the regularizer, such that the loss wrt. given labeled data is zero.
- The **separability problem** is a decision problem. It consists in deciding whether there exists a function in the family with loss zero for which the regularizer does not exceed a given bound.