Machine Learning II

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https://mlcv.cs.tu-dresden.de/courses/26-summer/ml2/

Summer Term 2026

Definition 1. A strict order on a set A is a relation $< \subseteq A \times A$ such that:

$$\forall a \in A\colon \quad \neg\, a < a \tag{1}$$
$$\forall a \in A\ \forall b \in A \setminus \{a\}\colon \quad a < b \ \text{ xor } \ b < a \tag{2}$$
$$\forall a \in A\ \forall b \in A \setminus \{a\}\ \forall c \in A \setminus \{a, b\}\colon \quad a < b \ \wedge \ b < c \ \Rightarrow \ a < c \tag{3}$$

Lemma 1. The strict orders on A are characterized by the bijections $\alpha : \{0, \ldots, |A| - 1\} \to A$.

Proof (sketch). For any such bijection, consider the relation $<_\alpha$ such that

$$\forall a, b \in A\colon \quad a < b \ \Leftrightarrow \ \alpha^{-1}(a) < \alpha^{-1}(b) \ . \tag{4}$$

Notation: For any set A, let $S_A := \{ab \in A \times A \mid a \neq b\}$.

Lemma 2. The strict orders on A are characterized by those $y \in \{0, 1\}^{S_A}$ that satisfy the following conditions:

$$\forall a \in A\ \forall b \in A \setminus \{a\}\colon \quad y_{ab} + y_{ba} = 1 \tag{5}$$
$$\forall a \in A\ \forall b \in A \setminus \{a\}\ \forall c \in A \setminus \{a, b\}\colon \quad y_{ab} + y_{bc} - 1 \leq y_{ac} \tag{6}$$

Definition 2 (Grötschel, 1984; see also Martí and Reinelt, 2011)**.** An instance of the **minimum cost linear ordering problem** is a pair (A, c) with A a finite set and $c \in \mathbb{R}^{S_A}$. Its feasible set is

$$Y_A := \{y \in \{0,1\}^{S_A} \mid \text{(5) and (6)}\} \ . \tag{7}$$

Its solutions are those $y^* \in Y_A$ with

$$\langle c, y^* \rangle = \min\{\langle c, y \rangle \mid y \in Y_A\} \ . \tag{8}$$

An instance of the **linear ordering problem** is a triple (A, c, k) with A a finite set, $c \in \mathbb{R}^{S_A}$ and $k \in \mathbb{R}$. Its solution set is

$$\mathsf{LOP}(A, c, k) = \{y \in Y_A \mid \langle c, y \rangle \leq k\} \ . \tag{9}$$

Theorem 1. The linear ordering decision problem is NP-complete.

It is obviously in NP. A well-known proof of its NP-hardness is by reduction of the feedback arc set problem:

Definition 3. For any digraph $G = (V, E)$ and any $F \subseteq E$, the set F is called a **feedback arc set** of G iff the graph $(V, E \setminus F)$ is acyclic.

Definition 4. An instance of the **feedback arc set problem** is a triple (V, E, m) such that $G = (V, E)$ is a digraph and $m \in \mathbb{N}_0$. Its solution set is

$$\mathsf{FAS}(V, E, m) = \{F \subseteq E \mid F \text{ is a feedback arc set of } (V, E) \text{ and } |F| \leq m\} \tag{10}$$

Theorem 2 (Karp 1972)**.** The feedback arc set problem is NP-complete.

Proof of Theorem 1 (sketch): Given an instance (V, E, m) of the feedback arc set problem, consider the instance (V, c, k) of the linear ordering problem with $k := m - |E|$ and $c \in \mathbb{R}^S$ such that

$$\forall ab \in S_V : \quad c_{ab} := \begin{cases} -1 & \text{if } ab \in E \\ 0 & \text{otherwise} \end{cases} \quad . \tag{11}$$

For any $y \in \{0,1\}^{S_V}$:

$$E = (E \cap y^{-1}(0)) \,\dot\cup\, (E \cap y^{-1}(1)) \ . \tag{12}$$

($\Rightarrow$) Let $y \in \mathsf{LOP}(V, c, k)$ arbitrary. $y^{-1}(1)$ is acyclic. Therefore, $y^{-1}(1) \cap E$ is acyclic. Thus, by (12), $E \setminus (y^{-1}(0) \cap E)$ is acyclic. Furthermore,

$$|E \cap y^{-1}(0)| \overset{(12)}{=} |E| - |E \cap y^{-1}(1)| = |E| + \langle c, y \rangle \leq |E| + (m - |E|) = m$$

Thus, $(y^{-1}(0) \cap E) \in \mathsf{FAS}(V, E, m)$.

($\Rightarrow$) Let $F \in \mathsf{FAS}(V, E, m)$ arbitrary. There exists a linear order and corresponding y such that $(E \cap y^{-1}(1)) = E \setminus F$. Therefore,

$$\langle c, y \rangle = -|E \cap y^{-1}(1)| = -|E \setminus F| = -|E| + |F| \leq -|E| + m = k \ . \tag{13}$$

Thus, $y \in \mathsf{LOP}(V, c, k)$. $\square$

Notation:

- For any $r \in \mathbb{R}$, let $r^{\pm} := \max\{0, \pm r\}$.
- For any instance (A, c) of the linear ordering problem and any $ab \in S_A$, let $c^{\Delta}_{ab} := c_{ab} - c_{ba}$.

Theorem 3. For any instance (A, c) of the linear ordering problem and any $ab \in S_A$ such that

$$c^{\Delta}_{ab} + \sum_{d \in A \setminus \{a,b\}} (c^{\Delta}_{ad} + c^{\Delta}_{db})^{+} \leq 0 \ , \qquad (14)$$

there exists an optimal solution y^* to this instance such that $y^*_{ab} = 1$.

We prove Theorem 3 by showing that, under the premise of the theorem, the map defined below is improving for the instance (A, c) of the linear ordering problem.

Definition 5. For any finite set A and any $\{a, b\} \in \binom{A}{2}$, define **transposition** $\tau_{\{a,b\}} \colon Y_A \to Y_A$ such that for all $y \in Y_A$ and all $a'b' \in S_A$:

$$\tau_{\{a,b\}}(y)_{a'b'} = \begin{cases} 1 - y_{a'b'} & \text{if } \{a', b'\} = \{a, b\} \\ y_{a'a} & \text{if } b' = b \wedge a' \neq a \\ y_{ab'} & \text{if } a' = b \wedge b' \neq a \\ y_{bb'} & \text{if } a' = a \wedge b' \neq b \\ y_{a'b} & \text{if } b' = a \wedge a' \neq b \\ y_{a'b'} & \text{if } \{a, b\} \cap \{a', b'\} = \varnothing \end{cases} . \qquad (15)$$

Define **conditional transposition** $\tau^1_{\{a,b\}} \colon Y_A \to Y_A$ such that for all $y \in Y_A$:

$$\tau^1_{ab}(y) = \begin{cases} x & \text{if } y_{ab} = 1 \\ \tau_{\{a,b\}}(y) & \text{otherwise} \end{cases} . \qquad (16)$$

To begin with, we calculate the effect of transposition on the objective value:

Lemma 3. Let (A, c) be any instance of the linear ordering problem, let $y \in Y_A$, let $\{a, b\} \in \binom{A}{2}$, and let $y' = \tau_{\{a,b\}}(y)$. Then:

$$\langle c, y' \rangle - \langle c, y \rangle = c^{\Delta}_{ab}(1 - 2y_{ab}) + \sum_{d \in A \setminus \{a,b\}} (c^{\Delta}_{ad} + c^{\Delta}_{db})(y_{bd} + y_{da} - 1) \ . \quad (17)$$

Proof.

$$\langle c, y'\rangle - \langle c, y\rangle = \sum_{a'b' \in S_A} c_{a'b'}(y'_{a'b'} - y_{a'b'}) = \sum_{\substack{a'b' \in S_A \\ \{a',b'\}\cap\{a,b\}\neq\varnothing}} c_{a'b'}(y'_{a'b'} - y_{a'b'})$$

$$= c_{ab}(y'_{ab} - y_{ab}) + c_{ba}(y'_{ba} - y_{ba})$$

$$+ \sum_{b' \in A\setminus\{a,b\}} c_{ab'}(y'_{ab'} - y_{ab'}) + \sum_{a' \in A\setminus\{a,b\}} c_{a'a}(y'_{a'a} - y_{a'a})$$

$$+ \sum_{b' \in A\setminus\{a,b\}} c_{bb'}(y'_{bb'} - y_{bb'}) + \sum_{a' \in A\setminus\{a,b\}} c_{a'b}(y'_{a'b} - y_{a'b})$$

$$= c_{ab}(1 - 2y_{ab}) + c_{ba}(1 - 2y_{ba})$$

$$+ \sum_{d \in A\setminus\{a,b\}} c_{ad}(y_{bd} - y_{ad}) + \sum_{d \in A\setminus\{a,b\}} c_{da}(y_{db} - y_{da})$$

$$+ \sum_{d \in A\setminus\{a,b\}} c_{bd}(y_{ad} - y_{bd}) + \sum_{d \in A\setminus\{a,b\}} c_{db}(y_{da} - y_{db})$$

$$= (c_{ab} - c_{ba})(1 - 2y_{ab}) + \sum_{d \in A\setminus\{a,b\}} (y_{da} + y_{bd} - 1)(c_{ad} - c_{da} - c_{bd} + c_{db})$$

$$= c^{\Delta}_{ab}(1 - 2y_{ab}) + \sum_{d \in A\setminus\{a,b\}} (y_{da} + y_{bd} - 1)(c^{\Delta}_{ad} + c^{\Delta}_{db})$$

□

Next, we establish a basic property of linear orders:

Lemma 4. For any finite set A, any $y \in Y_A$ such that $y_{ab} = 1$, and any $c \in A \setminus \{a, b\}$:

$$1 \leq y_{ac} + y_{cb} \quad . \tag{18}$$

Proof. For any $c \in A \setminus \{a, b\}$, we have

$$y_{bc} + y_{ca} - 1 \leq y_{ba} \quad .$$

On the left-hand side,

$$y_{bc} + y_{ca} - 1 = (1 - y_{cb}) + (1 - y_{ac}) - 1 = 1 - y_{ac} - y_{cb} \quad .$$

On the right-hand-side,

$$y_{ba} = 1 - y_{ab} = 1 - 1 = 0 \quad .$$

Thus:

$$1 \leq y_{ac} + y_{cb} \quad .$$

□

Proof of Theorem 3. Let y be any solution to the instance of the linear ordering problem, and let $y' = \tau^1_{ab}(y)$. If $y_{ab} = 1$ then $y' = y$. Thus, $\langle c, y' \rangle = \langle c, y \rangle$. If $y_{ab} = 0$ then

$$
\begin{aligned}
& \langle c, y' \rangle - \langle c, y \rangle \\
&= c^\Delta_{ab} + \sum_{d \in A \setminus \{a,b\}} (y_{da} + y_{bd} - 1)(c^\Delta_{ad} + c^\Delta_{db}) && \text{by Lemma 3} \\
&\leq c^\Delta_{ab} + \max_{\substack{x \in y_A \\ y_{ab}=0}} \sum_{d \in A \setminus \{a,b\}} (y_{da} + y_{bd} - 1)(c^\Delta_{ad} + c^\Delta_{db}) \\
&\leq c^\Delta_{ab} + \max_{\substack{x \in Y_A \\ y_{ab}=0}} \sum_{d \in A \setminus \{a,b\}} (y_{da} + y_{bd} - 1)(c^\Delta_{ad} + c^\Delta_{db})^+ && \text{by Lemma 4} \\
&= c^\Delta_{ab} + \sum_{d \in A \setminus \{a,b\}} (c^\Delta_{ad} + c^\Delta_{db})^+ && \text{as } y_{da} + y_{bd} - 1 \leq 1 \\
&\leq 0 && \text{by the premise} .
\end{aligned}
$$

Therefore, τ^1_{ab} is improving. Thus, y' is a solution to the instance (A, c) of the linear ordering problem. Moreover, $y'_{ab} = 1$. □

Definition 6. A **preorder** on a set A is a relation $\lesssim \subseteq A \times A$ such that:

$$\forall a \in A\colon \quad a \lesssim a \tag{19}$$

$$\forall a \in A \; \forall b \in A \setminus \{a\} \; \forall c \in A \setminus \{a,b\}\colon \quad a \lesssim b \;\wedge\; b \lesssim c \;\Rightarrow\; a \lesssim c \tag{20}$$

Example 1. On any set, the equivalence relations are the symmetric preorders, and the partial orders are the anti-symmetric preorders.

Remark 1. The symmetric subset $\sim$ of a preorder $\lesssim$ is an equivalence relation. The asymmetric subset $<$ of a preorder $\lesssim$ is a strict partial order. These relations are said to be induced by $\lesssim$.

Lemma 5. The strict partial order $<$ induced by a preorder $\lesssim \subseteq A \times A$ is well-defined also on $A/{\sim}$.

Proof. We need to show: $a, a', b, b' \in A$:

$$a < b \wedge a \sim a' \wedge b \sim b' \quad \Rightarrow \quad a' < b' \ . \tag{21}$$

Firstly,

$$\begin{aligned} & a \sim a' \wedge a < b \\ \Rightarrow\ & a' \lesssim a \wedge a \lesssim b \\ \Rightarrow\ & a' \lesssim b \end{aligned}$$

Secondly,

$$\begin{aligned} & a' \lesssim b \wedge b \sim b' \\ \Rightarrow\ & a' \lesssim b \wedge b \lesssim b' \\ \Rightarrow\ & a' \lesssim b' \ . \end{aligned}$$

Suppose now that $b' \lesssim a'$. Then $b \lesssim b' \lesssim a' \lesssim a$, in contradiction to $a < b$. Therefore,

$$b' \not\lesssim a' \ .$$

Thus, $a' < b'$. □

Lemma 6. The preorders on A are characterized by those $y \in \{0,1\}^{S_A}$ that satisfy

$$\forall a \in A \; \forall b \in A \setminus \{a\} \; \forall c \in A \setminus \{a,b\} \colon \quad y_{ab} + y_{bc} - 1 \leq y_{ac} \tag{22}$$

Definition 7. An instance of the **minimum cost preordering problem** is a pair (A, c) with A a finite set and $c \in \mathbb{R}^{S_A}$. Its feasible set is

$$Y_A := \{y \in \{0,1\}^{S_A} \mid (22)\} \ . \tag{23}$$

Its solutions are those $y^* \in Y_A$ with

$$\langle c, y^* \rangle = \min\{\langle c, y \rangle \mid y \in Y_A\} \ . \tag{24}$$

An instance of the **preordering problem** is a triple (A, c, k) with A a finite set, $c \in \mathbb{R}^{S_A}$ and $k \in \mathbb{R}$. Its solution set is

$$\mathsf{POP}(A, c, k) = \{y \in Y_A \mid \langle c, y \rangle \leq k\} \ . \tag{25}$$

Example 2. In the instance of the minimum cost preordering problem depicted below, edges indicate costs assigned to ordered pairs. Edges now shown indicate zero cost.