MACHINE LEARNING 1 WS2019/20
1. EXERCISE

- Probabilities -

problem 1. Bayes' theorem
Which of the following probabilities is sufficient to calculate $p(A|J,M)$?
There are no informations about conditional (in)dependence.

(1) p(J,M), p(A), p(J|A), p(M|A)
(2) p(J,M), p(A), p(J,M|A)
(3) p(A), p(J|A), p(M|A)

Solution: (2)

If J and M are conditional independend given A: which of the three tupel of probabilities is/are sufficient?

Solution: (1),(2) and (3) (cause p(J,M) can then be calculated by marginalization.)

problem 2. Two people want to meet in a period of length T. Everyone is coming equally likely somewhen in this period. The person arriving first waits for the second for a period of $a < T$. Determine the probability, that the two persons meet.

Solution: Consider the times of arrival t_1 and t_2 of the two persons. All points (t_1, t_2) in the square of period T are then equally likely. The function $f(t_1, t_2) = (|t_1 - t_2| < a)$ defines the area of that square where the two persons meet. Draw the square and the function in that square to get $p(|t_1 - t_2| < a) = \frac{(T^2 - (T-a)^2)}{T^2}$.

problem 3. A person claims to be able to always win in roulette. The strategy is thereby the following:
I bet on red with just one chip. If I win, I get two chips and leave the casino. If I lose, I bet again on red but this time with two chips. If I lose again, I bet again on red but with four chips. I double the bet until I win. At the end I get one chip more as I bet in total before. I always have 1023 chips with me. It means that I lose only if black wins 10 times in turn. This is however quite unlikely. Prove that the above claim is wrong by computing the expected profit for the proposed strategy.

Solution: see Wiki: Martingale (betting system)

problem 4. When tossing standard dice the probabilty distribution is $(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ for the values $(1, 2, 3, 4, 5, 6)$ on the die.

(1) If you toss 5 dice, what is the distribution for the maximum value, i.e. what are the probabilities to get a 1, 2, 3, 4, 5 or 6 as the maximum?

Solution:
If we consider any cumulative probability function $F(x)$ then this means that the probabilty to observe $v \leq x$ is equal to $F(x)$.
If we have $k = 5$ independent events then the probabilty to observe all $v_i \leq x$ is equal to $(F(x))^k$.
For our case of 5 dice from $p = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ we get $F = (\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6})$ and $F^5 = ((\frac{1}{6})^5, (\frac{2}{6})^5, (\frac{3}{6})^5, (\frac{4}{6})^5, (\frac{5}{6}^5), (\frac{6}{6})^5)$ and therefore the probabilities to observe the respective maximum values to be

$$p_{max} = \left(\left(\frac{1}{6}\right)^5, \left(\frac{2}{6}\right)^5 - \left(\frac{1}{6}\right)^5, \left(\frac{3}{6}\right)^5 - \left(\frac{2}{6}\right)^5, \left(\frac{4}{6}\right)^5 - \left(\frac{3}{6}\right)^5, \left(\frac{5}{6}\right)^5 - \left(\frac{4}{6}\right)^5, 1 - \left(\frac{5}{6}\right)^5\right)$$

$$= (0.0001286, \quad 0.0039866, \quad 0.02713477, \quad 0.100437, \quad 0.27019, \quad 0.598122)$$

But (in Theory) we could also count the different cases
(Thanks to those who worked out this solution)

$$P(max(x_1, \ldots, x_5) = x) = \sum_{i=1}^{5} \binom{5}{i} \cdot \left(\frac{x-1}{6}\right)^{(5-i)} \cdot \frac{1}{6^i}$$

The maximum value can be diced 1 to 5 times. Otherwise something lower.

$$P(\text{diced number} < x) = \left(\frac{x-1}{6}\right) \text{(4 to 0 times)}$$

$P(\text{diced number} = \text{x}) = \frac{1}{6}$ (1 to 5 times)
Because order does not matter all $\binom{5}{i}$ combinatorial possibilities have to be considered.

(2) Write a simulation program to verify your result.

Solution:

```
"""
5 dices
"""
import random
maxlist = [0,0,0,0,0,0]
nb_runs = 1000000
for k in range(nb_runs):
        max_value = 0
        for i in range(5):
                die = random.randint(1,6)
                if max_value < die:
                        max_value = die
        maxlist[max_value-1] +=1
print ("prop:", [x/nb_runs for x in maxlist])
```

(random) RESULT:
prop: [0.000129, 0.003907, 0.027211, 0.100335, 0.270197, 0.598221]

problem 5. You sample (without replacement) two locations (l_1, l_2) of a chess board. Let $c_1(l_1)$ and $c_2(l_2)$ be the random variables for the colors of the corresponding location. You can then consider e.g. the probability $p(c_1(l_1) = black, c_2(l_2) = white)$.
Given your sample is in a 8-neighborhood (here now called 'pair'), is the distribution $p(c_1(l_1), c_2(l_2)|pair(l_1, l_2))$ conditionally independent?

Solution: If

$$p(c_1(l_1), c_2(l_2) \,|\, pair(l_1, l_2))$$

is conditionally independent it has to be equal to

$$p(c_1(l_1) \,|\, pair(l_1, l_2)) \cdot p(c_2(l_2) \,|\, pair(l_1, l_2))$$

which is $(\frac{1}{4})$ in all cases because of symmetry. But looking at the border of the chess board we do have more unequal colored neighbors than equal ones - therefore this does not hold and this distribution cannot be conditionally independent. As shown in the exercise it is also possible to calculate all 8 probabiltity values of $(p(c_1(l_1), c_2(l_2) \,|\, pair(l_1, l_2)))$ und use these to get the same result.